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The Laminar Boundary Layer with Injection Through a Permeable Wall*

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SUMMARY

The steady incompressible laminar boundary layer equations for a perfect gas with arbitrary distributions of wall temperature, main stream velocity and normal velocity at the wall are solved approximately by a method similar to that of Lighthill (1950). Equations for the skin friction and the rate of heat transfer are obtained.

In order to assess the accuracy of these equations, solutions are presented and compared with exact solutions for wedge type flow with a wall temperature distribution such that the difference between the wall and the stream temperatures is proportional to a power of the distance from the leading edge. A Prandtl number of 0 7 is used. A solution for the skin friction with a constant normal velocity at the wall and constant main stream velocity is also presented.

The method is probably accurate enough for engineering purposes in regions of negative pressure gradient.

A solution for compressible flow using a similar method is outlined.

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CONTENTS

LIST OF SYMBOLS

a_1 , a_2 , A	Constants (see equations 8.8, 8.18, and 4.18 respectively)
b_1 , B, B_1	Constants (see equations 8.8, 4.18, and 7.37 respectively)
c	Constant defined by $u_1 = c\,x^m$
C_p	Specific heat at constant pressure
D	Constant (see equations 7.40 and 7.41)
E	Constant defined by equation 7.31
f_w	Dimensionless measure of the flow through the wall (see equation 7.6)
f_w''	Dimensionless skin friction (see equation 7.7)
$F_1(x)$	$= \int_0^x -\frac{2}{\mu}\,\tau_w(x')\,v_w(x')\,dx' + u_1^2(x)$
$F_2(x)$	$= -\frac{2\,\tau_w(x)}{\mu\rho}$
$F_3(x)$	$= -\frac{\sigma\,v_w(x)\,Q_w(x)}{\tau_w(x)\quad 2k}$
$F_4(x)$	$= \frac{1}{k}\left(\frac{\mu}{2\,\rho\,\tau_w(x)}\right)^{\frac{1}{2}} Q_w(x)$
G	Constant defined by equation 7.3
$G(x\Phi)$	$= Z(x, \Phi) + \int_0^x S(x', \Phi)\,dx'$
$I_\nu(-)$	Modified Bessel function of first kind
k	Thermal conductivity
K, K_1	Constants defined by equations 7.4 and 8.17 respectively
$K_\nu(-)$	Modified Bessel function of second kind
L	Length of plate
m	Euler number $\quad -x\,\frac{dp}{dx} \Big/ \rho u_1^2$; $u_1 = c\,x^m$

List of Symbols (Continued)

M	Constant defined by equation 7.27
$N_{n-1,n-2}$	Defined by equation 5.2
nu	Local Nusselt number
Nu	Overall Nusselt number
p	Pressure; Heaviside operator
P	Function of Φ in equation 4.12
q	$= \frac{4}{3} p^{\frac{1}{2}} \quad \Phi^{\frac{3}{4}}$, where p is Heaviside operator
Q	Function of Φ in equation 4.12
$Q_w(x)$	Rate of heat transfer to the wall
$R,\ R_1$	Constants (see equations 8.7 and 8.18 respectively)
Re	Reynolds number
$S(x, \Phi)$	$= \left(\frac{\partial Z}{\partial \Phi}\right)_x \quad \left(\frac{\partial \Phi}{\partial x}\right)_\psi$
t	Independent variable (see equations 3.8 and 4.6a); t_1 dummy variable of integration
T	Temperature (A temperature scale with the free stream zero)
$\overline{T}_1\ ,\ \overline{T}_2$	Defined by equation 4.13
u	Velocity in x-direction; u_1 is free stream velocity
v	Velocity in y-direction
$W\left\{ - \right\}$	Wronskian
x	Co-ordinate from stagnation point along the plate; x_1 , and x' , dummy variables of integration
y	Co-ordinate normal to the plate
Z	$u_1^2 - u^2$
z	Dummy variable in lieu of x

List of Symbols (Continued)

α	Defined by equation 7.32
β	Exponent from $\tau_w(x) = K^2 x^\beta$
γ	Exponent from $v_w(x) = Gx^\gamma$
$\Gamma(-)$	Gamma function
ϵ	Exponent from $T_w(x) = B_1 x^\epsilon$; $\epsilon = \frac{x}{T_w} \cdot \frac{dT_w}{dx}$
ξ	$= \frac{v_w^2 x}{u_1 \nu}$
ρ	Density
μ	Dynamic viscosity
ν	Kinematic viscosity
σ	Prandtl number, $\frac{C_p \mu}{k}$
ψ	Stream function
Φ	Modified stream function, $\psi - \psi_w$
$\tau_w(x)$	Wall shear stress

Subscripts

w	Refers to conditions at the wall
n	Refers to the n^{th} iteration
$\overline{(-)}$	The 'bar' denotes a Laplace transform (see equ. 3.10a)

1. Introduction

In order to predict the skin friction or the heat transfer at the surface of a body moving through the atmosphere, it is essential to know the behaviour of the boundary layer over the body. It may be necessary to heat a surface to prevent icing or alternatively to cool the surface to prevent it reaching high temperatures. The high temperatures at high Mach numbers are due to the heat transferred from the boundary layer which is heated by viscous stresses. An effective method of cooling heated bodies is to inject a gas through a permeable wall into the boundary layer, thus modifying the velocity and temperature profiles and thereby reducing the heat transfer to the surface. Suction through a permeable wall may be used to control the boundary layer by increasing the skin friction and delaying separation.

Many solutions to the boundary layer equations have been reported for flow over a flat plate which has a normal velocity at the surface. Fage and Falkner (Ref. 1) solved the problem for conditions of constant fluid properties, variable wall temperature and a linear velocity increase normal to the wall. Emmons and Leigh (Ref. 2) used the Blasius transformation and obtained numerical solutions to the laminar boundary layer equations for the conditions of constant wall temperature and a normal velocity at the wall which is proportional to the inverse half power of the distance from the leading edge. These assumptions lead to 'similar' velocity and temperature profiles. Brown and others (Refs. 3, 4 and 5) used a modified Blasius transformation and obtained solutions for a wedge-type flow (flow for which the main stream velocity is proportional to a power of the distance from the stagnation point) with variable fluid properties and constant wall temperature. Donoughe and Livingood (Ref. 6) used a similar method for wedge-type flow with constant fluid properties but a variable wall temperature. In the numerical solutions for the wedge-type flow, the normal velocity at the wall is limited to one of the form:

a constant x(tne distance from the stagnation point)$^{\frac{m-1}{2}}$

where m is the Euler number.

Iglisch (Ref. 7) obtains exact numerical solutions when there is constant suction along the wall. Approximate results for constant suction are presented by Curle (Ref. 8) who extends a method used by Stratford (Ref. 22) in which the pressure forces in the boundary layer are equated to the viscous forces.

A little experimental work has been performed for the low speed cases by Duwes and Wheeler (Ref. 9), Libby and others (Ref. 10) and by Mickley and others (Ref. 11).

Solutions of the compressible laminar boundary layer including the effects of transpiration cooling have been presented by Low (Ref. 12) who uses a method similar to that of Chapman and Rubesin (Ref. 13). Yuan (Ref. 14) has assumed that a polynomial of the fourth degree may be used to represent the velocity profile and solves the problem by a Kármán-Pohlhausen method. Lew and Fanucci (Refs. 15 and 16) have used a modified Kármán momentum method and also an exact method to solve the equations.

In the solutions which have been obtained so far, the normal velocity at the wall has been severely limited by the transformations used. In this paper an extension of Lighthill's method (Ref. 17) has been used to obtain an approximate solution to the incompressible laminar boundary layer with arbitrary distributions of wall temperature, of main stream velocity and of normal velocity at the wall. Two integral equations are obtained for the skin friction and the heat transfer rate to the wall. In order to estimate the accuracy of these integral equations, solutions are obtained for the particular case of wedge-type flow. The solutions are then compared with the exact results of Donoughe and Livingood (Ref.6). A further solution for the skin friction is obtained for the case of uniform suction at the wall and is then compared with the exact results of Iglisch (Ref. 7). The accuracy of the integral equations is of the same order as that of Lighthill's equations which were for an impermeable wall.

The uses of an incompressible solution are extremely limited but the method has been extended to the compressible case by using a paper by Lilley (Ref. 18). The results are presented in Appendix B. The stability of the laminar boundary layer under conditions of flow injection is not considered.

2. The Boundary Layer Equations with Flow through the Porous Wall

Referring to Fig. 1, the subscript $_1$ denotes conditions outside the boundary layer and the subscript $_w$ denotes conditions at the wall. For a perfect gas the equations of continuity, momentum and energy for a steady laminar incompressible boundary layer flow in two dimensions are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\cdot\frac{\partial p}{\partial x} \tag{2.2}$$

and

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\mu}{\rho\sigma}\cdot\frac{\partial^2 T}{\partial y^2} \tag{2.3}$$

The Von Mises' transformation which changes the independent variables (x, y) to (x, ψ), where ψ is the stream function, is given by

$$\frac{\partial \psi}{\partial x} = -\rho v \quad \text{and} \quad \frac{\partial \psi}{\partial y} = \rho u \tag{2.4}$$

These relations satisfy the continuity equation, 2.1. On applying the transformation to the momentum and energy equations then

$$\frac{\partial Z}{\partial x} = \mu\rho u \frac{\partial^2 Z}{\partial \psi^2} \tag{2.5}$$

and

$$\frac{\partial T}{\partial x} = -\frac{\mu\rho}{2C_p}\cdot\frac{\partial u}{\partial \psi}\cdot\frac{\partial Z}{\partial \psi} + \frac{1}{\sigma}\cdot\frac{\partial}{\partial \psi}\left(\mu\rho u\frac{\partial T}{\partial \psi}\right). \tag{2.6}$$

$$\text{where } Z = u_1^2(x) - u^2(x\,\psi). \tag{2.7}$$

A modified stream function $\Phi = \psi - \psi_w$ (2.8)

is now introduced.
ψ_w is defined by $\left(\frac{\partial \psi}{\partial x}\right)_{y=0} = -\rho v_w = \frac{\partial \psi w}{\partial x}.$ (2.9)

Hence, using the independent variables (x, Φ), equations 2.5 and 2.6 may be written

$$\left(\frac{\partial Z}{\partial \Phi}\right)_x \left(\frac{\partial \Phi}{\partial x}\right)_\psi + \left(\frac{\partial Z}{\partial x}\right)_\Phi = \mu \rho u \left(\frac{\partial^2 Z}{\partial \Phi^2}\right)_x \tag{2.10}$$

and

$$\left(\frac{\partial T}{\partial \Phi}\right)_x \left(\frac{\partial \Phi}{\partial x}\right)_\psi + \left(\frac{\partial T}{\partial x}\right)_\Phi = \frac{1}{\sigma} \cdot \frac{\partial}{\partial \Phi} \left(\mu \rho u \frac{\partial T}{\partial \Phi}\right)_x \tag{2.11}$$

In equation 2.11 the frictional heating term has been assumed negligible and has been omitted. Lighthill (Ref. 17) obtained approximate solutions to these equations when the velocity at the wall, v_w, was zero. Lighthill's method was extended by Bernard Le Fur (Ref. 19) to include the frictional heating term.

3. The Solution of the Momentum Equation (2.10)

In order to solve equation 2.10, a method very similar to that of Lilley (Ref. 18) will now be used.

Let

$$\left(\frac{\partial Z}{\partial \Phi}\right)_x \left(\frac{\partial \Phi}{\partial x}\right)_\psi = S(x, \Phi) \tag{3.1}$$

Now $S(x, \Phi) = \frac{\partial}{\partial x} \int_0^x S(x', \Phi)dx'$, so that equation 2.10 may be written:

$$\frac{\partial}{\partial x} \left(Z + \int_0^x S(x', \Phi)dx'\right) = \mu \rho u \frac{\partial^2}{\partial \Phi^2} \left(Z + \int_0^x S(x', 0)dx'\right). \tag{3.2}$$

At $\Phi = 0$ then $\int_0^x S(x', \Phi)dx' = \int_0^x S(x', 0)dx'$ and

at $\Phi = \infty$ then $\int_0^x S(x', \Phi)dx' = 0$ since $S(x, \infty) = 0$.

Also $\frac{\partial^2}{\partial \Phi^2} \int_0^x S(x', \Phi)dx' = 0$ at $\Phi = \infty$.

Hence, near $\Phi = 0$, equation 3.2 may be written approximately as

$$\frac{\partial}{\partial x} \cdot G(x, \Phi) = \mu \rho u \frac{\partial^2}{\partial \Phi^2} \cdot G(x, \Phi) \tag{3.3}$$

$$\text{where } G(x, \Phi) = Z(x, \Phi) + \int_0^x S(x', 0)dx'$$

For larger values of Φ it has a similar form with $G(x, \Phi) = Z + \int_0^x S(x', \Phi)dx'$ such that $G(x, \infty) = 0$.

Following Page and Falkner (Ref. 1), an expression for the velocity u, namely $u = \frac{\tau_W(x) . y}{\mu}$, which is most closely accurate near the wall, will now be substituted into the equations.

From equation 2.4:

$$\psi = \rho \int_0^y u\, dy = \rho \frac{\tau_W(x)}{2\mu} . y^2 + \psi_W(x)$$

and thus

$$u = \left(\frac{2\, \tau_W(x)}{\mu\rho} \Phi \right)^{\frac{1}{2}} \quad \text{as } \Phi \to 0 . \tag{3.4}$$

On substituting 3.4 into equation 3.3, then

$$\frac{\partial G}{\partial x} = \left(2\, \mu\rho\, \tau_W(x) \right)^{\frac{1}{2}} \Phi^{\frac{1}{2}} \frac{\partial^2 G}{\partial \Phi^2} . \tag{3.5}$$

From equations 3.4 and 2.7

$$\left(\frac{\partial Z}{\partial \Phi} \right)_x = - \frac{2\, \tau_W(x)}{\mu\rho} \quad \text{as } \Phi \to 0 . \tag{3.6};$$

Using this equation together with 2.9 and 3.1 then

$$\int_0^x S(x', 0)dx' = \int_0^x - \frac{2\, \tau_W(x')}{\mu} . v_W(x')\, dx' .$$

The boundary conditions for $G(x, \Phi)$, for which 3.3 is to be solved, are therefore

$$\Phi = \infty \quad G(x, \Phi) = 0 ,$$

$$x \to 0 \quad G(x, \Phi) \to 0 ,$$

$$\Phi \to 0 \quad G(x, \Phi) = u_1^2(x) + \int_0^x - \frac{2\,\tau_W(x')}{\mu} \cdot v_W(x')dx' - \frac{2\,\tau_W(x)}{\mu\rho}\,\Phi$$

$$= F_1(x) + F_2(x) .\ \Phi . \qquad (3.7)$$

The functions $F_1(x)$ and $F_2(x)$ are defined by equation 3.7.

If $$t = \int_0^x \left(2\mu\rho\,\tau_W(z)\right)^{\frac{1}{2}} dz \qquad (3.8)$$

then equation 3.5 may be written

$$\frac{\partial G}{\partial t}(t, \Phi) = \Phi^{\frac{1}{2}} . \quad \frac{\partial^2 G}{\partial \Phi^2}(t, \Phi) . \qquad (3.9)$$

The boundary conditions at $\Phi = 0$ are

$$G = F_1(t) \quad \text{and} \quad \frac{\partial G}{\partial \Phi} = F_2(t) . \qquad (3.10)$$

Using the Laplace transform notation $\overline{F}(p, \Phi) = \int_0^\infty e^{-pt} F(t, \Phi) dt$ (3.10a)

and the condition $\overline{G} = 0$ when $t = 0$, equation 3.9 may be written

$$p\,\overline{G} = \Phi^{\frac{1}{2}} \quad \frac{\partial^2 \overline{G}}{\partial \Phi^2} . \qquad (3.11)$$

Equation (66) of Lighthill's paper (Ref. 17) is analogous to equation 3.11 and both satisfy similar boundary conditions.
The solution of equation 3.11 is thus

$$\overline{G} = (\tfrac{2}{3})^{\frac{2}{3}}\, p^{\frac{1}{3}}\, \Phi^{\frac{1}{2}} .\ \Gamma(\tfrac{1}{3}) .\ I_{-\frac{2}{3}}(\tfrac{4}{3}\, p^{\frac{1}{2}}\, \Phi^{\frac{3}{4}})\, \overline{F}_1 + (\tfrac{2}{3}\, p^{\frac{1}{2}})^{-\frac{2}{3}}\, \Phi^{\frac{1}{2}}\, \Gamma(\tfrac{5}{3}) .$$
$$I_{\frac{2}{3}}(\tfrac{4}{3}\, p^{\frac{1}{2}}\, \Phi^{\frac{3}{4}})\, \overline{F}_2 \qquad (3.12)$$

where $I_{-\frac{2}{3}}$ and $I_{\frac{2}{3}}$ are modified Bessel functions. Now $\overline{G} \to 0$ as $\Phi \to \infty$ therefore, from the properties of Bessel functions, the coefficients of $I_{-\frac{2}{3}}$ and $I_{\frac{2}{3}}$ must be equal and opposite. Hence

$$\overline{F}_1 = -(\tfrac{2}{3})^{-\frac{1}{3}}\, p^{-\frac{2}{3}}\, \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \cdot \overline{F}_2 \qquad (3.13)$$

or, taking the inverse transform

$$F_1(t) = -(\tfrac{2}{3})^{-\frac{1}{3}} \frac{1}{\Gamma(\frac{1}{3})} \int_0^t \frac{F_2(t_1)}{(t - t_1)^{\frac{1}{3}}} dt_1 \quad .$$

Using equation 3.8 and substituting for $F_1(t)$ and $F_2(t)$ from equation 3.7 the following equation is obtained after simplifying:

$$u_1^2(x) = \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3}).(\mu\rho)^{\frac{2}{3}}} \int_0^x \left(\int_{x_1}^x \tau_w(z)^{\frac{1}{2}} dz \right)^{-\frac{1}{3}} \tau_w(x_1)^{3/2} dx_1 + \frac{2}{\mu} \int_0^x v_w(x_1).\ \tau_w(x_1)\, dx_1 \quad . \tag{3.14}$$

Solutions to this integral equation are obtained in Section 7.1.

4. The Solution of the Energy Equation 2.11

A method similar to that used by Lighthill (Ref. 17) will be used to find the heat transfer to the wall in terms of an arbitrary wall temperature and skin friction distribution and a non-zero velocity, at the wall.

In finding the approximate solution to the momentum equation in the previous section, conditions which were accurate at the wall were substituted for overall conditions. This procedure will again be adopted. The approximation for u, equation 3.4, will again be used.

The rate of heat transfer to the wall is given by

$$Q_w(x) = \left(k \frac{\partial T}{\partial y} \right)_w \quad . \tag{4.1}$$

Now

$$\left(\frac{\partial T}{\partial \Phi} \right)_x = \left(\frac{\partial T}{\partial y} \right)_x \left(\frac{\partial y}{\partial \Phi} \right)_x \quad . \tag{4.2}$$

From the approximation for u,

$$\left(\frac{\partial y}{\partial \Phi} \right)_x = \left(\frac{\mu}{\rho\, 2\, \tau_w(x)} \right)^{\frac{1}{2}} \Phi^{-\frac{1}{2}} \quad . \tag{4.3}$$

Hence, from the above three equations together with 2.9

$$\left(\frac{\partial T}{\partial \Phi}\right)_x \left(\frac{\partial \Phi}{\partial x}\right)_\psi = \frac{Q_w(x)}{k} \left(\frac{\mu}{\rho 2 \tau_w(x)}\right)^{\frac{1}{2}} \Phi^{-\frac{1}{2}} \rho v_w(x).$$

On substituting this into the energy equation 2.11 then

$$\frac{Q_w(x)}{k} \left(\frac{\mu}{\rho 2 \tau_w}\right)^{\frac{1}{2}} \Phi^{-\frac{1}{2}} \rho v_w + \left(\frac{\partial T}{\partial x}\right)_\Phi =$$

$$\frac{1}{\sigma}(4 \tau_w \mu \rho)^{\frac{1}{2}} \frac{\partial}{\partial \Phi}\left(\Phi^{\frac{1}{2}} \frac{\partial T}{\partial \Phi}\right) . \qquad (4.4)$$

Let $$F_3(x) = - \frac{\sigma v_w Q_w}{\tau_w 2 k} \qquad (4.5)$$

then equation 4.4 simplifies to:

$$\Phi^{-\frac{1}{2}} F_3 = \left(\frac{\partial T}{\partial x}\right)_\Phi \frac{\sigma}{(2 \mu \rho \tau_w)^{\frac{1}{2}}} - \frac{\partial}{\partial \Phi}\left(\Phi^{\frac{1}{2}} \frac{\partial T}{\partial \Phi}\right) . \qquad (4.6)$$

Now if $$t = \int_0^x \frac{1}{\sigma}(2 \mu \rho \tau_w)^{\frac{1}{2}} dx_1, \qquad (4.6a)$$

(this is different from the t used in Section 3) equation 4.6 may be written

$$\Phi^{-\frac{1}{2}} F_3 = \left(\frac{\partial T}{\partial t}\right)_\Phi - \frac{\partial}{\partial \Phi}\left(\Phi^{\frac{1}{2}} \frac{\partial T}{\partial \Phi}\right) . \qquad (4.7)$$

The boundary conditions are:

(a) $T = 0$ at $t = 0$ and at $\Phi = \infty$,

(b) $T = T_w(t)$ at $\Phi = 0$ and, (4.8)

(c) $\left(k \frac{\partial T}{\partial y}\right)_w = Q_w(t)$ at $\Phi = 0$.

Using the transform notation of equation 3.10a equation 4.7 becomes

$$p \bar{T} - \frac{\partial}{\partial \Phi}\left(\Phi^{\frac{1}{2}} \frac{\partial \bar{T}}{\partial \Phi}\right) = \Phi^{-\frac{1}{2}} \bar{F}_3 . \qquad (4.9)$$

The homogeneous form of this equation is equivalent to equation (21) in Lighthill's paper (Ref. 17) and the solution is

$$\bar{T} = a \Phi^{\frac{1}{4}} I_{-\frac{1}{3}}\left(\frac{4}{3} p^{\frac{1}{2}} \Phi^{\frac{3}{4}}\right) + b \Phi^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4}{3} p^{\frac{1}{2}} \Phi^{\frac{3}{4}}\right) \qquad (4.10)$$

where a and b are determined from the boundary conditions.

The complete solution of the equation 4.9 will now be found by the method of variation of parameters.

Let $$q = \frac{4}{3} p^{\frac{1}{2}} \Phi^{\frac{3}{4}} . \tag{4.11}$$

A solution of 4.9 is assumed to be

$$\overline{T} = P(\Phi)\, \overline{T}_1 + Q(\Phi)\, \overline{T}_2 \tag{4.12}$$

where $\overline{T}_1 = \Phi^{\frac{1}{4}} I_{-\frac{1}{3}}(q)$

and $$\overline{T}_2 = \Phi^{\frac{1}{4}} I_{\frac{1}{3}}(q) . \tag{4.13}$$

The equations to determine P and Q, are thus

$$\frac{dP}{d\Phi} = - \frac{\overline{T}_2 \Phi^{-1} \overline{F}_3(p)}{\overline{T}_1' \overline{T}_2 - \overline{T}_2' \overline{T}_1}$$

and

$$\frac{dQ}{d\Phi} = \frac{\overline{T}_1 \Phi^{-1} \overline{F}_3(p)}{\overline{T}_1' \overline{T}_2 - \overline{T}_2' \overline{T}_1} \tag{4.14}$$

where the prime indicates a differential with respect to Φ .

From 4.13

$$\overline{T}_1' \overline{T}_2 - \overline{T}_2' \overline{T}_1 = \Phi^{\frac{1}{2}} \left\{ I_{\frac{1}{3}}(q)\, I'_{-\frac{1}{3}}(q) - I_{-\frac{1}{3}}(q).\, I'_{\frac{1}{3}}(q) \right\}.$$

Now the Wronskian, $W(I_{\frac{1}{3}}(\xi), I_{-\frac{1}{3}}(\xi)) = - \frac{2}{\xi\pi} \operatorname{Sin} \frac{\pi}{3}$.

Hence $$\overline{T}_1' \overline{T}_2 - \overline{T}_2' \overline{T}_1 = - \Phi^{\frac{1}{2}} \left\{ \frac{3}{2\pi} \cdot \frac{1}{\Phi} \operatorname{Sin} \frac{\pi}{3} \right\}. \tag{4.15}$$

The equations 4.14 may now be written:

$$P = \int^{\Phi} \left(\frac{\overline{F}_3\, 2\pi}{3 \operatorname{Sin} \pi/3} \right) I_{\frac{1}{3}}(q)\, \Phi^{-\frac{1}{4}} d\Phi \tag{4.16}$$

and

$$Q = - \int^{\Phi} \left(\frac{\overline{F}_3\, 2\pi}{3 \operatorname{Sin} \pi/3} \right) I_{-\frac{1}{3}}(q)\, \Phi^{-\frac{1}{4}} d\Phi . \tag{4.17}$$

The solution to equation 4.9 is obtained by substituting 4.16 and 4.17 into equation 4.12, thus:

$$\overline{T}(p\,\Phi) = \Phi^{\frac{1}{4}} I_{-\frac{1}{3}}(q) \left(\frac{2\,\pi\,\overline{F}_3}{3\,\mathrm{Sin}\,\pi/3}\right) \int_0^{\Phi} I_{\frac{1}{3}}(q)\ \Phi^{-\frac{1}{4}}\ d\Phi\ -$$

$$\Phi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \left(\frac{2\pi\,\overline{F}_3}{3\,\mathrm{Sin}\,\pi/3}\right) \int_0^{\Phi} I_{-\frac{1}{3}}(q)\ \Phi^{-\frac{1}{4}}\ d\Phi\ + A\Phi^{\frac{1}{4}} I_{-\frac{1}{3}}(q)\ + B\,\Phi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \qquad (4.18)$$

where A and B are to be found from the boundary conditions 4.8. They are not functions of Φ .

As $\Phi \to \infty$ then $\overline{T} \to 0$, therefore the coefficients of $I_{\frac{1}{3}}$ and $I_{-\frac{1}{3}}$ must be equal and opposite. Hence:

$$\frac{2\pi\ \overline{F}_3\ p^{-\frac{1}{2}}}{3\ \mathrm{Sin}\,\pi/3} \int_0^{\infty} \left(I_{\frac{1}{3}}(q) - I_{-\frac{1}{3}}(q)\right) dq + A + B = 0\ ,$$

or

$$-\ \frac{4}{3}\ \ \overline{F}_3\ p^{-\frac{1}{2}} \int_0^{\infty} K_{\frac{1}{3}}(q)\ dq + A + B = 0\ . \qquad (4.19)$$

$K_{\frac{1}{3}}(q)$ is a modified Bessel function of the second kind.

From Ref. 20,

$$\int_0^{\infty} K_{\nu}\ (\beta c)\ c^{\alpha - 1}\ dc\ =\ 2^{\alpha - 2}\ \ \beta^{-\alpha}\Gamma(\tfrac{1}{2}\,\alpha + \tfrac{1}{2}\,\nu)\Gamma(\tfrac{1}{2}\,\alpha - \tfrac{1}{2}\,\nu)$$

providing $(\alpha \mp \nu) > 0$ and $\beta > 0$.

Thus equation 4.19 may be written

$$-\ \tfrac{2}{3}\ \Gamma(\tfrac{2}{3})\ \Gamma(\tfrac{1}{3})\ p^{-\frac{1}{2}}\ \ \overline{F}_3\ + A + B\ =\ 0\ . \qquad (4.20)$$

From equation 4.18, as $\Phi \to 0$ then

$$\overline{T}(p\Phi)\ \to\ A(\frac{3}{2})^{\frac{1}{3}}\ \ \frac{p^{-1/6}}{\Gamma(\frac{2}{3})}\ . \qquad (4.21)$$

Thus by using the boundary condition 4.8(b)

$$A\ =\ (\tfrac{2}{3})^{\frac{1}{3}}\ \ \Gamma(\tfrac{2}{3})\ p^{1/6}\ \ \overline{T}_w(p)\ . \qquad (4.22)$$

Now from equation 4.3 and the boundary condition 4.8(c),

$$k\ \left(\frac{\partial T}{\partial y}\right)\ =\ k\ \left(\frac{\partial T}{\partial \Phi}\right) \Phi^{\frac{1}{2}}\ \left(\frac{2\ \rho\ \tau_w(t)}{\mu}\right)^{\frac{1}{2}}\ =\ Q_w(t)\ \text{ at } \Phi\ = 0.$$

F_4 will be defined by the equation

$$F_4(t) = \left(\frac{\partial T}{\partial \Phi}\right) \Phi^{\frac{1}{2}} = \frac{1}{k} \left(\frac{\mu}{2 \rho \tau_w}\right)^{\frac{1}{2}} Q_w(t). \tag{4.23}$$

On differentiating equation 4.18 with respect to Φ and considering the limit as $\Phi \to 0$ then

$$\Phi^{\frac{1}{2}} \frac{\partial \overline{T}}{\partial \Phi} \to B \left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{p^{1/6}}{\Gamma(\frac{1}{3})} .$$

Therefore from 4.23

$$B = p^{-1/6} \Gamma(\tfrac{1}{3}). (\tfrac{2}{3})^{\frac{2}{3}} \overline{F}_4(p) . \tag{4.24}$$

After substituting equation 4.24 and 4.22 into 4.20 :

$$\overline{F}_4 = (\tfrac{2}{3})^{\frac{1}{3}} \Gamma(\tfrac{2}{3}). p^{-\frac{1}{3}} \overline{F}_3 - (\tfrac{3}{2})^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} p^{\frac{1}{3}} \overline{T}_w . \tag{4.25}$$

By the Convolution Theorem the inverse transform of $p^{\frac{1}{3}} \overline{T}_w$ is given by

$$p^{\frac{1}{3}} \overline{T}_w(p) \sqsubset \frac{1}{\Gamma(\frac{2}{3})} \int_o^t \frac{\left(\frac{\partial T_w(t_1)}{\partial t_1} + \delta(t_1) T_w(t_1)\right)dt_1}{(t - t_1)^{\frac{1}{3}}} \tag{4.26}$$

where δ is the Delta function (an impulse function). For shortness equation 4.26 will be written in Stieltjes form. Using equation 4.6a then

$$p^{\frac{1}{3}} \overline{T}_w(p) \sqsubset \frac{1}{\Gamma(\frac{2}{3})} \left(\frac{2 \mu \rho}{\sigma^2}\right)^{-1/6} \int_o^x \left(\int_{x'}^x \tau_w(z)^{\frac{1}{2}}. dz\right)^{-\frac{1}{3}} d\left[T_w(x')\right]. \tag{4.27}$$

Similarly

$$p^{-\frac{1}{3}} \overline{F}_3(p) \sqsubset - \frac{1}{\Gamma(\frac{1}{3})} \left(\frac{2 \mu \rho}{\sigma^2}\right)^{1/6} \left(\frac{\sigma}{2k}\right) \int_o^x \left(\int_{x'}^x \tau_w(z)^{\frac{1}{2}} dz\right)^{-\frac{2}{3}} \frac{v_w(x') Q_w(x') dx'}{\tau_w(x')^{\frac{1}{2}}} . \tag{4.28}$$

Using 4.27 and 4.28, the inverse transform of equation 4.25 is

$$Q_w(x) = \left[Q_w(x)\right]_{v_w(x)=0} - \left(\frac{\sigma^2 \rho^2}{3 \mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \cdot \tau_w(x)^{\frac{1}{2}} \int_0^x \left(\int_{x'}^x \tau_w(z)^{\frac{1}{2}} dz\right)^{-\frac{2}{3}} \frac{v_w(x') Q_w(x')dx'}{\tau_w(x')^{\frac{1}{2}}} \qquad (4.29)$$

where $\left[Q_w(x)\right]_{v_w=0}$ is the heat transfer to the wall when the velocity, $v_w(x)$, at the wall is zero. It is the expression which was obtained by Lighthill (1950) and is given by

$$\left[Q_w(x)\right]_{v_w=0} = -k \left(\frac{3\sigma\rho}{\mu^2}\right)^{\frac{1}{3}} \frac{\tau_w(x)^{\frac{1}{2}}}{\Gamma(\frac{1}{3})} \int_0^x \left(\int_{x'}^x \tau_w(z)^{\frac{1}{2}} dz\right)^{-\frac{1}{3}} d\, T_w(x') \; . \qquad (4.30)$$

.....

After changing the order of integration the total heat transfer rate over an area of surface between x = 0 and L is given by :

$$\int_0^L Q_w(x)\, dx = \int_0^L \left[Q_w(x)\right]_{v_w=0} dx - \left(\frac{9\, \sigma^2 \rho^2}{\mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \int_0^L \left(\int_{x'}^L \tau_w(z)^{\frac{1}{2}} dz\right)^{\frac{1}{3}} . \frac{v_w(x') Q_w(x')dx'}{\tau_w(x')^{\frac{1}{2}}} \qquad (4.31)$$

Equation 4.29 with equation 4.30 is a Volterra type integral equation and can be solved by a standard iterative procedure (Ref. 24). However it was found more convenient in the present case to use a slightly modified procedure and this is explained below.

5. A General Equation for the Heat Transfer Rate $Q_w(x)$

Let $(Q_w)_n$ be the expression for Q_w after the n^{th} iteration. The equation for $(Q_w)_n$ is obtained directly from equation 4.29.

$$(Q_w)_n = (Q_w)_{n-1} - \left(\frac{\rho^2 \sigma^2}{3 \mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \tau_w^{\frac{1}{2}} \int_0^x \left(\int_{x'}^x \tau_w^{\frac{1}{2}} dz\right)^{-\frac{2}{3}} \frac{v_w}{\tau_w^{\frac{1}{2}}} N_{n-1,n-2} dx' \qquad (5.1)$$

where $N_{n-1,n-2} = (Q_w)_{n-1} - (Q_w)_{n-2}$. (5.2)

On integrating equation 5.1 from 0 to L and changing the order of integration then

$$\int_o^L (Q_w)_n \, dx = \int_o^L (Q_w)_{n-1} \, dx - \left(\frac{\rho^2 \sigma^2 g}{\mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_o^L \left(\int_{x'}^L \tau_w^{\frac{1}{2}} dz\right)^{\frac{1}{3}}$$

$$\frac{v_w}{\tau_w^{\frac{1}{2}}} \cdot N_{n-1,n-2} \, dx' . \tag{5.3}$$

This equation will be used in Sections 7.4 and 7.5.

6. <u>The Nusselt Number</u>

An overall Nusselt number, Nu, may be defined by

$$Nu = - \int_o^L \frac{Q_w \, dx}{k \, T_m} \tag{6.1}$$

where T_m is a mean wall temperature.

A Local Nusselt number, nu, is defined by

$$nu = - \frac{Q_w \, x}{k \, T_w} \tag{6.2}$$

(where x is the distance from the leading edge).

Equation 5.1 may therefore be written:

$$(nu)_n = (nu)_{n-1} - \left(\frac{\rho^2 \sigma^2}{3 \, \mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \frac{x \, \tau_w^{\frac{1}{2}}}{k \, T_w} \int_o^x \left(\int_{x'}^x \tau_w^{\frac{1}{2}} \, dz\right)^{-\frac{2}{3}} \frac{v_w}{\tau_w^{\frac{1}{2}}}$$

$$\left\{ \frac{k \, T_w}{x'} \left((nu)_{n-1} - (nu)_{n-2} \right) \right\} dx' \tag{6.3}$$

This equation will be used in Section 7.6.

7. Solution of the Integral Equations when the Wall Shear Stress $\tau_w(x) = \text{Const.} \times x^\beta$ and the Free Stream Velocity $u_1(x) = cx^m$

7.1. The momentum equation

The initial assumptions are that

$$u_1(x) = c\,x^m \tag{7.1}$$

and $\qquad \tau_w(x) = K^2 x^\beta \quad . \qquad$ (m, β, c and K are constants).

On substituting equations 7.1 into equation 3.14 and integrating:

$$c^2 x^{2m} = \frac{2 \cdot 3^{\frac{1}{3}} K^{8/3}}{\Gamma(\frac{4}{3}) \cdot (\mu\rho)^{\frac{2}{3}}} \left(\frac{2}{\beta + 2} \right)^{\frac{2}{3}} x^{\frac{4\beta + 2}{3}} \frac{\Gamma\left(\frac{3\beta+2}{\beta+2}\right)\Gamma(\frac{2}{3})}{\Gamma\left(\frac{3\beta+2}{\beta+2} + \frac{2}{3}\right)} + \frac{2}{\mu} \int_0^x v_w K^2 x_1^\beta \, dx_1 \ . \tag{7.2}$$

It is obvious from this equation that the expression for $v_w(x)$ is restricted to one of the form

$$v_w(x) = G\,x^\gamma \tag{7.3}$$

where γ is determined from 7.2.

Hence $\quad \gamma = \dfrac{m - 1}{2} \quad$ and $\quad \beta = \dfrac{3m - 1}{2}$

providing $\quad \beta + \gamma \neq -1$.

The assumptions 7.1 therefore imply that

$$\begin{aligned} u_1(x) &= c\,x^m \ , \\ \tau_w(x) &= K^2\,x^{\frac{3m-1}{2}} \ , \\ v_w(x) &= G\,x^{\frac{m-1}{2}} \ . \qquad m \neq 0 \ . \end{aligned} \tag{7.4}$$

These conditions, excluding the condition $m \neq 0$, are the same as those used by Brown and Donoughe (Ref. 4), Emmons and Leigh (Ref. 2) and Donoughe and Livingood (Ref. 6). (They are the conditions for similarity between the velocity and temperature profiles when $\sigma = 1$).

Equations 7.2 can now be written

$$c^2 = \frac{2.3^{\frac{1}{3}} K^{8/3}}{(\mu\rho)^{\frac{2}{3}}} \left(\frac{4}{3m+3}\right)^{\frac{2}{3}} \frac{\Gamma(\frac{2}{3})\,\Gamma\left(\frac{9m+1}{3m+3}\right)}{\Gamma(\frac{1}{3})\Gamma\left(\frac{11m+3}{3m+3}\right)} + \frac{K^2 G}{\mu\, m} . \qquad (7.5)$$

The function, f_w (a dimensionless measure of the coolant flow through the porous wall) and the function, f''_w (the dimensionless skin friction), are now introduced.

$$f_w = -\frac{2}{m+1} v_w(x) \left(\frac{x}{u_1(x)\nu}\right)^{\frac{1}{2}} \qquad (7.6)$$

and

$$f''_w = \tau_w(x) \left(\frac{x}{\rho\, u_1^3(x)\,\mu}\right)^{\frac{1}{2}} . \qquad (7.7)$$

Hence equation 7.5 is

$$1 + \frac{(m+1)}{2m} f_w \cdot f''_w = \frac{2^{7/3}}{3^{\frac{1}{3}}(m+1)^{\frac{2}{3}}} \cdot \frac{\Gamma(\frac{2}{3})\,\Gamma\left(\frac{9m+1}{3m+3}\right)}{\Gamma(\frac{1}{3})\,\Gamma\left(\frac{11m+3}{3m+3}\right)} (f''_w)^{4/3} . \qquad (7.8)$$

When $m = 0.5$ then equation 7.8 is

$$1 + 1.5\, f_w \cdot f''_w = 1.284\, (f''_w)^{4/3} . \qquad (7.9)$$

When $m = 1.0$ then equation 7.8 is

$$1 + f_w\, f''_w = 0.843\, (f''_w)^{4/3} . \qquad (7.10)$$

f''_w in equation 7.9 and 7.10 has been evaluated for certain values of f_w. The results are presented in Table 1, and are compared with the exact results presented by Donoughe and Livingood (Ref. 6). The errors are very large ranging from 8% to 24%. The reason for the large error is that it contains the combined errors of both terms on the right hand side of equation 3.14.

However an improved solution for f''_w can be obtained if we assume that equation 7.8 is only used to determine the difference between $(f''_w)_{v_w=0}$ and f''_w where $(f''_w)_{v_w=0}$ is that calculated by Lighthill, or

$$\Delta(f''_w) = (f''_w)_{\text{equ. } 7.8} - (f''_w)_{\text{Lighthill}} \tag{7.11}$$

When this difference is obtained the absolute value of f''_w can be found by using the exact values for $(f''_w)_{v_w=0}$ as calculated by Hartree (Ref. 21), or

$$f''_w = (f''_w)_{\text{Hartree}} + \Delta f''_w \tag{7.12}$$

7.2. <u>A partial inversion of the momentum equation 3.14</u>

Equation 3.14 may be written (see Appendix A)

$$\tau_w(x) = \frac{(\rho\mu)^{\frac{2}{3}}}{\Gamma(\frac{2}{3}).\ 3^{\frac{1}{3}}} \left\{ -\frac{1}{\mu} \int_o^x \left(\int_{x'}^x \tau_w^{\frac{1}{2}}\, dz \right)^{-\frac{2}{3}} v_w \tau_w \, dx' + \frac{1}{2} \int_o^x \left(\int_{x'}^x \tau_w^{\frac{1}{2}}\, dz \right)^{-\frac{2}{3}} d \left[u_1^2(x') \right] \right\} . \tag{7.13}$$

If expressions for τ_w, u_1 and v_w are substituted into 7.13 then the equation obtained will be the same as that obtained from 7.5. The reason for writing equation 3.13 in the above form is to investigate the effect of introducing the approximation

$$\int_{x'}^{x} \tau_w(z)^{\frac{1}{2}}\, dz \simeq (x - x')\, \tau_w^{\frac{1}{2}}(x') , \tag{7.14}$$

which Lighthill (1950) suggested.

This approximation will only be used in evaluating the term containing $v_w(x)$.

The expressions for $\tau_w(x)$, $u_1(x)$ and $v_w(x)$ of equations 7.4 are substituted into the integrals of equation 7.13. Thus:

$$\frac{1}{2}\int_{o}^{x}\left(\int_{x'}^{x}\tau_w^{\frac{1}{2}}\,dz\right)^{-\frac{2}{3}}.\ d(u_1^2(x')) = \tfrac{1}{2}\,c^2\left(\frac{3(m+1)}{4K}\right)^{\frac{2}{3}} x^{\frac{3m-1}{2}}.\ \frac{\Gamma(\frac{1}{3})\,\Gamma\left(\frac{11m+3}{3(m+1)}\right)}{\Gamma\left(\frac{9m+1}{3m+3}\right)} \tag{7.15}$$

and on using the approximation 7.14 for the term which contains $v_w(x)$ then

$$-\frac{1}{\mu}\int_{o}^{x}\left(\int_{x'}^{x}\tau_w^{\frac{1}{2}}\,dz\right)^{-\frac{2}{3}} v_w\ \tau_w\,dx' \simeq -\frac{K^{4/3}G}{\mu}\,x^{\frac{3m-1}{2}}\ \frac{\Gamma(\frac{1}{3})\Gamma\left(\frac{9m+1}{6}\right)}{\Gamma\left(\frac{3m+1}{2}\right)}\ . \tag{7.16}$$

If equations 7.15 and 7.16 are substituted into 7.13 then:

$$\left(f_w''\right)^{4/3} = \frac{(m+1)\,\Gamma(\frac{1}{3})\Gamma\left(\frac{9m+1}{6}\right)}{3^{\frac{1}{3}}.2.\Gamma(\frac{2}{3})\Gamma\left(\frac{3m+1}{2}\right)}\ f_w.\ f_w'' + (m+1)^{\frac{2}{3}}.\ \frac{3^{\frac{1}{3}}\Gamma(\frac{1}{3})\,\Gamma\left(\frac{11m+3}{3(m+1)}\right)}{2^{7/3}\Gamma(\frac{2}{3})\ \Gamma\left(\frac{9m+1}{3(m+1)}\right)}\ \ \ldots.. \tag{7.17}$$

When m = 0 then this equation reduces to

$$\left(f_w''\right)^{4/3} = 0.211 + 2.15\ f_w.\ f_w''\ . \tag{7.18}$$

When m = 0.5 then

$$\left(f_w''\right)^{4/3} = 0.779 + 1.20\ f_w\ f_w''\ . \tag{7.19}$$

When m = 1.0 then

$$\left(f_w''\right)^{4/3} = 1.19 + 1.24\ f_w.\ f_w''\ . \tag{7.20}$$

The value of f_w'' is modified as in Section 7.1., equation 7.12.
The results are presented in Figs. 2 and 3.

7.3. The first iteration of the energy equation

The total heat transfer rate over an area between $x = 0$ and $x = L$ is given by equation 4.31 in terms of Q_w. An iterative procedure will be used and the first approximation for Q_w will be Lighthill's expression (equation 4.30).

The condition of constant wall temperature, T_w (or constant difference between the free stream temperature and the wall temperature) will be considered.

Expressions for $\tau_w(x)$, $u_1(x)$ and $v_w(x)$ from equations 7.4 and $Q_w(x)$ from 4.30 are substituted into equation 4.31 and, after integrating :

$$\int_0^L (Q_w)_1 dx = -\frac{3\,k\,T_w}{2\,\Gamma(\frac{1}{3})} \left(\frac{3\sigma\rho}{\mu^2}\right)^{\frac{1}{3}} \left(\frac{4\,K}{3(m+1)}\right)^{\frac{2}{3}} L^{\frac{m+1}{2}} + 2\,T_w k \left(\frac{\sigma\rho}{\mu}\right) G.\,L^{\frac{m+1}{2}}. \quad \ldots\ldots \quad (7.21)$$

On introducing the overall Nusselt number, Nu, defined by equation 6.1 :

$$(Nu)_{\text{First iteration}} = \frac{3^{4/3}}{2.\,\Gamma(\frac{1}{3})}\,\sigma^{\frac{1}{3}}.\,\left[f_w''\right]^{\frac{1}{3}}\left[\tfrac{3}{4}(m+1)\right]^{-\frac{2}{3}} - \sigma f_w\,. \quad (7.22)$$

On substituting numerical values it is found that this equation over-estimates the correction required. More iterations will be necessary.

7.4. The second iteration of the energy equation

$N_{n-1,n-2}(x)$ is defined by equation 5.2, such that

$$N_{1,0}(x) = (Q_w)_1 - (Q_w)_{v_w=0}\ .$$

Therefore from equation 7.21

$$\int_0^x N_{1,0}(x)\,dx_1 = \frac{2\,T_w k\,\sigma\,\rho\,G}{(m+1)\,\mu}\,x^{\frac{m+1}{2}} \quad (7.23)$$

and

$$N_{1,0}(x) = \frac{\sigma\rho}{\mu} G\,x^{\frac{m-1}{2}}\,.\,T_w.\,k. \quad (7.24)$$

Proceeding as before :

$$\int_0^L (Q_w)_2\, dx = \int_0^L (Q_w)_1\, dx - \frac{\rho^{5/3}\,\sigma^{5/3}}{\mu^{4/3}} \left(\frac{4}{3(m+1)}\right)^{4/3} \frac{T_w .\ k\, G^2\, \Gamma(\frac{1}{3})\, 3^{\frac{2}{3}}}{2\, K^{\frac{2}{3}}}\ L^{\frac{m+1}{2}}, \tag{7.25}$$

and

$$(Nu)_{\text{Second iteration}} = 0.807.\ \sigma^{\frac{1}{3}} \left[f''_w\right]^{\frac{1}{3}} \left(\tfrac{3}{4}(m+1)\right)^{-\frac{2}{3}} - \sigma f_w + \frac{1.02\ \sigma^{5/3} (m+1)^{\frac{2}{3}}\, f_w^2}{\left[f''_w\right]^{\frac{1}{3}}} . \tag{7.26}$$

7.5. <u>Iterations using the general equation 5.3</u>

From the previous Sections 7.3 and 7.4, it is seen that $\int_0^x N_{n-1,n-2}\ dx'$ has the form

$$\int_0^x N_{n-1,n-2}(x')\, dx' = M\, x^{\frac{m+1}{2}} \tag{7.27}$$

where M is independent of x.

Equation 5.3 may now be written

$$\int_0^L (Q_w)_n\ dx = \int_0^L (Q_w)_{n-1}\ dx -$$

$$2\left(\frac{\rho^2\sigma^2}{3\mu}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{2}{3})\, T_w\, G\, M}{\Gamma(\frac{1}{3})\, K^{\frac{2}{3}}} \left(\frac{4}{3(m+1)}\right)^{-\frac{2}{3}} \int_0^L \left(L^{\frac{3(m+1)}{4}} - x_1^{\frac{3(m+1)}{4}}\right)^{\frac{1}{3}} x_1^{\frac{m-3}{4}}\ dx_1 .$$

On integrating and rearranging then

$$\left(Nu\, Re^{-\frac{1}{2}}\right)_n = \left(Nu\, Re^{-\frac{1}{2}}\right)_{n-1} - \left(\frac{mL^{\frac{m+1}{2}}}{k\, Re^{\frac{1}{2}}}\right) \frac{\Gamma(\frac{1}{3})}{2^{\frac{1}{3}}\, 3^{\frac{2}{3}}} .\ \sigma^{\frac{2}{3}} (m+1)^{\frac{2}{3}} f_w .\left[f''_w\right]^{-\frac{1}{3}}. \quad \ldots\ldots \tag{7.28}$$

From equation 7.27 and 5.2

$$\frac{M\,L}{k}^{\frac{m+1}{2}} = \int_0^L \left((Nu)_{n-2} - (Nu)_{n-1} \right) dx \ . \qquad (7.29)$$

Providing the value of $(Nu\ Re^{-\frac{1}{2}})$ when $v_w(x) = 0$ is known, all other terms in the series may be obtained from 7.28.

From equation 7.28 and 7.29

$$(Nu\ Re^{-\frac{1}{2}}) = (Nu\ Re^{-\frac{1}{2}})_{v_w(x)=0} \left\{ 1 + \frac{E\,f_w}{(f''_w)^{\frac{1}{3}}} + \left(\frac{E\,f_w}{(f''_w)^{\frac{1}{3}}}\right)^2 + \left(\frac{E\,f_w}{(f''_w)^{\frac{1}{3}}}\right)^3 \ldots\ldots \right\}$$

$$\ldots\ldots \qquad (7.30)$$

$$\text{where } E = \frac{\Gamma(\frac{1}{3})\ \sigma^{\frac{2}{3}}(m+1)^{\frac{2}{3}}}{2^{\frac{1}{3}}\ 3^{\frac{2}{3}}}, \qquad (7.31)$$

$$\text{or } Nu\ Re^{-\frac{1}{2}} = (Nu\ Re^{-\frac{1}{2}})_{v_w(x)=0} \left\{ \frac{1}{1 - \frac{Ef_w}{(f''_w)^{\frac{1}{3}}}} \right\},$$

$$\text{providing } -1 < \frac{E.\ f_w}{(f''_w)^{\frac{1}{3}}} < 1 \ .$$

Lighthill's expression for $(Nu\ Re^{-\frac{1}{2}})_{v_w(x)=0}$, from equation 4.30 is

$$(Nu\ Re^{-\frac{1}{2}})_{v_w(x)=0} = \frac{3^{\frac{2}{3}}\ 2^{\frac{1}{3}}(m+1)^{-\frac{2}{3}}\ \sigma^{\frac{1}{3}}\ \left[f''_w\right]^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}$$

After substituting this equation into 7.30 it is found that the first three terms are the same as those of equation 7.26.

7.6. The local Nusselt number, nu

By comparison with equation 7.27, a relation between the local Nusselt numbers after consecutive iterations is found to be

$$\frac{nu_{n-1} - nu_{n-2}}{x} = \alpha\, x^{\frac{m-1}{2}} \quad \text{where } \alpha \text{ is independent of } x. \qquad (7.32)$$

Therefore equation 6.3 simplifies to

$$(\text{nu Re}^{-\frac{1}{2}})_n = (\text{nu Re}^{-\frac{1}{2}})_{n-1} + \left(\frac{\alpha x^{\frac{m+1}{2}}}{\text{Re}^{\frac{1}{2}}}\right) \Gamma(\tfrac{1}{3})\ \sigma^{\frac{2}{3}} (m+1)^{\frac{2}{3}}\ \frac{f_w}{(f''_w)^{\frac{1}{3}}} \quad \ldots\ldots \quad (7.33)$$

Lighthill's expression for $(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0}$, from equation 4.30 is

$$(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0} = \frac{2^{-\frac{2}{3}}\ 3^{\frac{2}{3}}\ \sigma^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}\ (f''_w)^{\frac{1}{3}}\ (m+1)^{\frac{1}{3}}\ . \quad (7.34)$$

From equations 7.34 and 7.33, the series for the local Nusselt number is

$$\text{nu Re}^{-\frac{1}{2}} = (\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0} \left\{ 1 + \frac{E f_w}{(f''_w)^{\frac{1}{3}}} + \left(\frac{E f_w}{(f''_w)^{\frac{1}{3}}}\right)^2 \ldots\ldots \right\}$$

or

$$\text{nu Re}^{-\frac{1}{2}} = (\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0} \left(\frac{(f''_w)^{\frac{1}{3}}}{(f''_w)^{\frac{1}{3}} - E f_w}\right) \quad (7.35)$$

providing $\left| \frac{E f_w}{(f''_w)^{\frac{1}{3}}} \right| < 1$.

E is defined by equation 7.31.

In the equations for the overall and the local Nusselt numbers, the skin friction parameter, f''_w, must be evaluated at the appropriate f_w. Because $(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0}$ is proportional to $(f''_w)^{\frac{1}{3}}$ (from equation 7.34) the correction which is used to convert the first term of equation 7.35 to one having the correct skin friction parameter is

$$(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0} \left(\frac{(f''_w) \text{ At particular } f_w}{(f''_w) \text{ at } f_w = 0} \right)^{\frac{1}{3}}$$

so that equation 7.35 may be written:

$$\text{nu Re}^{-\frac{1}{2}} = \left(\frac{\text{nu Re}^{-\frac{1}{2}}}{(f''_w)^{\frac{1}{3}}}\right)_{v_w(x)=0} \times \left(\frac{(f''_w)^{\frac{2}{3}}}{(f''_w)^{\frac{1}{3}} - E f_w}\right) . \quad (7.36)$$

Values for nu $Re^{-\frac{1}{2}}$ are presented in Table 3. During the calculation the exact values obtained by Donoughe and Livingood (Ref. 6) were used for $(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0}$ and for f''_w .

7.7. Wall temperature variation $T_w = B_1 x^{\epsilon}$

Let the variation in T_w be $B_1 x^{\epsilon}$; B_1 and ϵ are constants. (7.37)

When this equation together with the equations 7.4 are substituted into equation 4.30 and the resulting equation is integrated and non-dimensionalised then

$$(\text{nu Re}^{-\frac{1}{2}})_{v_w(x)=0} = 2^{4/3} \sigma^{\frac{1}{3}} f''^{\frac{1}{3}}_w (m+1)^{-\frac{2}{3}} \frac{\epsilon\, \Gamma(\frac{2}{3})\, \Gamma\left(\frac{4\epsilon}{3(m+1)}\right)}{3^{\frac{1}{3}}\, \Gamma(\frac{1}{3})\, \Gamma\left(\frac{4\epsilon}{3(m+1)} + \frac{2}{3}\right)} \quad \ldots\ldots \quad (7.38)$$

By comparison with equation 7.32, the relation between the local Nusselt numbers after consecutive iterations is

$$T_w \left(\frac{nu_{n-1} - nu_{n-2}}{x} \right) = \alpha\, x^{\frac{m-1+2\epsilon}{2}} \cdot B_1 \quad (7.39)$$

If this is substituted into the general equation 6.3 then

$$(\text{nu Re}^{-\frac{1}{2}})_n = (\text{nu Re}^{-\frac{1}{2}})_{n-1} + \left(\frac{\alpha x^{\frac{m+1}{2}}}{Re^{\frac{1}{2}}}\right) \left(\frac{D.\, f_w}{(f''_w)^{\frac{1}{3}}}\right) \quad (7.40)$$

where

$$D = \frac{\sigma^{\frac{2}{3}} (m+1)^{\frac{2}{3}}\, \Gamma(\frac{2}{3})\, \Gamma\left(\frac{4\epsilon}{3(m+1)} + \frac{1}{3}\right)}{3^{\frac{2}{3}}\, 2^{\frac{1}{3}}\, \Gamma\left(\frac{4\epsilon}{3(m+1)} + \frac{2}{3}\right)} \quad (7.41)$$

Hence

$$(\text{nu Re}^{-\frac{1}{2}}) = \left(\frac{\text{nu Re}^{-\frac{1}{2}}}{f''^{\frac{1}{3}}_w}\right)_{v_w(x)=0} \left(\frac{f''^{\frac{1}{3}}_w}{1 - \frac{Df_w}{f''^{\frac{1}{3}}_w}}\right), \quad (7.42)$$

providing $\left| \frac{Df_w}{(f''_w)^{\frac{1}{3}}} \right| < 1$.

When $\epsilon = 0$ this equation is identical with 7.35. The numerical values for nu $Re^{-\frac{1}{2}}$ for various values of f_w and m are presented together with those of the previous section in Table 3 and also in Figs. 4, 5, and 6.

8. Approximate Solution to the Integral Equations when the Free Stream is Uniform and the Velocity, v_w is a constant

The partially inverted momentum equation which is derived in Appendix A is

$$\tau_w(x) = \frac{(\rho\mu)^{\frac{2}{3}}}{\Gamma(\frac{2}{3})\,3^{\frac{1}{3}}} \left\{ -\frac{1}{\mu}\int_o^x \left(\int_{x_1}^x \tau_w(z)^{\frac{1}{2}}\,dz\right)^{-\frac{2}{3}} v_w(x_1)\,\tau_w(x_1)\,dx + \frac{1}{2}\int_o^x \left(\int_{x_1}^x \tau_w(z)^{\frac{1}{2}}\,dz\right)^{-\frac{2}{3}} d\left[u_1^2(x_1)\right] \right\}. \qquad (8.1)$$

When m = 0 in the equations 7.4 and when $v_w = 0$, then u_1 = a constant

and $\tau_w(x) = K^2 x^{-\frac{1}{2}}$ (8.2)

where from equation 7.5

$$K = \frac{u_1^{\frac{3}{4}}\, 3^{1/8} (\mu\rho)^{\frac{1}{4}}}{2^{7/8}(\Gamma(\frac{2}{3}))^{3/8}}. \qquad (8.3)$$

Equation 8.2 will be used as a first approximation to $\tau_w(x)$. On substituting 8.2 into the right hand side of 8.1 and integrating then the second approximation for $\tau_w(x)$ becomes

$$(\tau_w)_{\text{2nd Approx.}} = \frac{(\rho\mu)^{\frac{2}{3}}}{\Gamma(\frac{2}{3}).\,3^{\frac{1}{3}}} \left\{ -\frac{v_w}{\mu} K^{4/3} \left(\frac{4}{3}\right)^{\frac{1}{3}}.\,\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) + \tfrac{1}{2}c^2 K^{-\frac{2}{3}} (\tfrac{3}{4})^{\frac{2}{3}} x^{-\frac{1}{2}} \right\}$$

..... (8.4)

Now the dimensionless form of the flow through the wall is defined by equation 7.6 as

$$f_w = -\frac{2\,v_w\,Re^{\frac{1}{2}}}{u_1} \qquad (8.5)$$

where $Re = \dfrac{u_1 x}{\nu}$

Thus equation 8.4 may be written

$$(\tau_w(x))_{\text{2nd Approx.}} = (1 - R x^{\frac{1}{2}}) K^2 x^{-\frac{1}{2}} \tag{8. 6}$$

where $Rx^{\frac{1}{2}} = - \dfrac{2^{\frac{1}{4}} \Gamma(\frac{1}{3}) \, \Gamma^{\frac{1}{4}}(\frac{2}{3})}{3^{\frac{3}{4}}} f_w$. R is not a function of x. (8.7)

If it were now possible to substitute equation 8.6 into the right hand side of equation 8.1 and integrate, then a third approximation for $\tau_w(x)$ would be found. If this procedure were continued then an equation for $\tau_w(x)$ of the form

$$\tau_w(x) = K^2 x^{-\frac{1}{2}} (1 - a_1 (Rx^{\frac{1}{2}}) - b_1 (Rx^{\frac{1}{2}})^2 \quad \ldots\ldots), \quad |Rx^{\frac{1}{2}}| < 1, \tag{8.8}$$

could be obtained.

In this paper only the first two terms will be considered. This implies that $Rx^{\frac{1}{2}}$ is small, i.e. the velocity through the wall is very small. The equation for $\tau_w(x)$ thus obtained will then be compared with those of Iglisch (Ref. 7) and Curle (Ref. 8).

In order to find the value of a_1 in equation 8 .8, this expression for $\tau_w(x)$ will be substituted into the terms on the right hand side of equation 8.1. Three different approximate methods will be used.

First Method

$\int_{x_1}^{x} \tau_w(z)^{\frac{1}{2}} dz$ will be put equal to

$$\frac{4}{3} \tau_w(x_1)^{\frac{1}{2}} x_1^{\frac{1}{4}} (x^{\frac{3}{4}} - x_1^{\frac{3}{4}}) \tag{8.9}$$

which is correct only when $v_w = 0$.

On substituting $\tau_w(x) = K^2 x^{-\frac{1}{2}} (1 - a_1 Rx^{\frac{1}{2}})$ (8.10)

into the integrals of equation 8.1 :

$$\int_0^x \left(\int_{x_1}^x \tau_w^{\frac{1}{2}} dz \right)^{-\frac{2}{3}} v_w \, \tau_w \, dx_1 \simeq v_w (\tfrac{4}{3})^{\frac{1}{3}} K^{4/3} \left\{ \Gamma(\tfrac{1}{3}) \, \Gamma(\tfrac{2}{3}) - (\tfrac{1}{3}) a_1 Rx^{\frac{1}{2}} \frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} \right\} \quad \ldots\ldots \tag{8.11}$$

and $$\frac{c^2}{2}\left(\int_0^x \tau_w(z)^{\frac{1}{2}}dz\right)^{-\frac{2}{3}} \simeq \frac{c^2}{2}\left(\tfrac{3}{4}\right)^{\frac{2}{3}} K^{-\frac{2}{3}} x^{-\frac{1}{2}} \left(1 + \tfrac{1}{3} a_1 Rx^{\frac{1}{2}} \ldots\right) \quad (8.12)$$

Hence on substituting 8.11 and 8.12 into 8.1

$$\tau_w(x) = K^2 x^{-\frac{1}{2}} \left(1 + (\tfrac{1}{3} a_1 - 1)Rx^{\frac{1}{2}} \ldots\right) . \quad (8.13)$$

Therefore from 8.8 and 8.13 $a_1 = \frac{3}{4}$ and so

$$\tau_w(x) = K^2 x^{-\frac{1}{2}} \left(1 - \tfrac{3}{4} Rx^{\frac{1}{2}} \ldots\right) . \quad (8.14)$$

Second Method

If the approximation which Lighthill (1950) suggested is used, i.e.

$$\int_{x_1}^{x} \tau_w(z)^{\frac{1}{2}} dz \simeq (x - x_1)\tau_w(x_1)^{\frac{1}{2}} , \quad (8.15)$$

then from equation 8.1 the value of $\tau_w(x)$ when $v_w(x) = 0$ is modified to $\tau_w(x) = K_1^2 x^{-\frac{1}{2}}$ (8.16)

where $$K_1 = \frac{u_1^{\frac{3}{4}} (\mu\rho)^{\frac{1}{4}}}{2^{3/8}(\Gamma(\frac{2}{3}))^{3/8}} . \quad (8.17)$$

Equation 8.10 is modified to

$$\tau_w(x) = K_1^2 x^{-\frac{1}{2}} \left(1 - a_2 (R_1 x^{\frac{1}{2}}) \ldots\right) , \quad (8.18)$$

where $$R_1 x^{\frac{1}{2}} = \frac{-\Gamma(\frac{1}{3})(\Gamma(\frac{2}{3}))^{\frac{1}{4}}}{2^{\frac{3}{4}}\ 3^{\frac{1}{4}}} . f_w \simeq -1.31 f_w . \quad (8.19)$$

On substituting 8.18 into 8.1 then the equation obtained is

$$\tau_w = K_1^2 x^{-\frac{1}{2}} \left(1 + \tfrac{1}{3}(a_2 - 1)R_1 x^{\frac{1}{2}} \ldots\right) .$$

Hence $$\tau_w = K_1^2 x^{-\frac{1}{2}} \left(1 - \tfrac{3}{4} R_1 x^{\frac{1}{2}} \ldots\right) . \quad (8.20)$$

Third Method

In the first and second methods the respective approximations 8.9 and 8.15 were used for both terms on the right hand side of 8.1. The second term can however be evaluated without using an approximation.

The expression which is obtained in place of 8.12 is thus :

$$\frac{c^2}{2}\left(\frac{4}{3}\right)^{-\frac{2}{3}} K^{-\frac{2}{3}} x^{-\frac{1}{2}} \left(1 + \frac{1}{5} a_1 R x^{\frac{1}{2}} \ldots\right)$$

and the final equation for $\tau_w(x)$ is given by

$$\tau_w(x) = K^2 x^{-\frac{1}{2}} \left(1 - \frac{5}{6} R x^{\frac{1}{2}} \ldots\right) . \tag{8.21}$$

In order to compare equations 8.14 8.20 and 8.21 with previous work, the function ξ is introduced such that

$$\xi = \frac{v_w^2 . x}{u_1 \ \nu} \quad . \tag{8.22}$$

Now from equation 8.7 $\quad Rx^{\frac{1}{2}} \simeq 3\xi^{\frac{1}{2}}$.

Similarly from 8.19 $\quad R_1 x^{\frac{1}{2}} \simeq 2.61\, \xi^{\frac{1}{2}}$.

Therefore equation 8.14 is

$$\frac{\tau_w(x)}{K^2 x^{-\frac{1}{2}}} = 1 + 2.25\, \xi^{\frac{1}{2}} , \qquad |\xi| < \tfrac{1}{3} \tag{8.23}$$

equation 8.20 is

$$\frac{\tau_w(x)}{K_1^2 x^{-\frac{1}{2}}} = 1 + 1.96\, \xi^{\frac{1}{2}}, \qquad |\xi| < .38 \tag{8.24}$$

and equation 8.21 is

$$\frac{\tau_w(x)}{K^2 x^{-\frac{1}{2}}} = 1 + 2.5\, \xi^{\frac{1}{2}}, \qquad |\xi| < \tfrac{1}{3} \tag{8.25}$$

These equations are presented in Fig. 7 and compared with the results of Curle (Ref. 8) and Iglisch (Ref. 7).

9. Discussion

Two approximate integral equations have been obtained for the skin friction and for the heat transfer rate to the wall. The approximation used in the momentum equation consisted of replacing the velocity u with its asymptotic form as $y \to 0$, and in the energy equation the velocity u and $\frac{\partial T}{\partial y}$ were replaced with their asymptotic forms as $y \to 0$. The argument is that the most important part of the velocity and temperature profiles are those parts near the wall, since the precise way in which they approach the main stream values will not greatly influence the wall conditions. The errors introduced by this method must be obtained by comparing the final equations with exact numerical solutions for special cases.

Lighthill (Ref. 17) only used the approximation $u = \frac{\tau_w(x)\, y}{\mu}$ and this is correct in the limit for a very large velocity boundary layer thickness and a very small temperature boundary layer thickness. In this paper the heat transfer equation with fluid injection at the wall is not asymptotic for large σ because the further approximation, $T = \frac{Q_w(x).\, y}{k}$ had to be introduced.

In the momentum equation only the approximation for u was used. If this had been substituted immediately into equation 2.10 then the resulting equation would not have satisfied the boundary condition at infinity. Because of this, a method very similar to that of Lilley (Ref. 18) was used. A function $S(x, \Phi)$ is introduced which by definition does satisfy the boundary conditions at $\Phi = \infty$ and at $\Phi = 0$. In this way the difficulty is overcome. The approximation for u has to be used in a term which is equivalent to that in Lighthill's paper and also in a term which includes the velocity of blowing or suction at the wall. It is therefore reasonable to expect that there will be a greater error in the final equation for the skin friction with a velocity, $v_w(x)$, at the wall.

The resulting equation (3.14) was compared with the exact solutions of Donoughe and Livingood (Ref. 6) for the particular case of wedge type flow. A singularity exists when the Euler number, m, is zero. This is probably due to the physically impossible condition of a constant free stream velocity being unaffected by an infinite velocity, v_w, which exists at the leading edge. The numerical values for the skin friction parameter f''_w, are presented in Table 1 and they show large errors. However in a practical application of the integral equation, the conditions when v_w is zero will probably be known. If this is so then the modified equation 7.11 may be used. The results are given in Table 2 and the errors are now less than 11%. In Section 7.2, f''_w, is evaluated by an approximate method in which there is no singularity at m = 0. The values for f''_w using this method, together with those of the previous method are presented in Figs. 2 and 3. For conditions of negative pressure gradient the results are reasonably good.

The equation for the local Nusselt number is obtained from the heat transfer integral equation for the particular case of wedge-type flow. The results in Table 3 and the curves of Figs. 4, 5 and 6 are calculated using the modified equation 7.42. The results are poor when m is zero but this probably reflects the singularity which existed in the skin friction equation. The local Nusselt number is reasonably accurate over the region for which the equation 7.42 is valid.

In the curves of Fig. 7 the equations predicted in this paper are only valid for $\xi \ll \frac{1}{3}$, therefore the curves are extended past this limit empirically.

An analysis similar to that of Section 8 for a flat plate with an impermeable portion from the leading edge to a point x_1, but with a constant velocity v_w thereafter has been considered. All the terms of order $(Rx^{\frac{1}{2}})^2$ and higher have been neglected and the final equation is:

$$\frac{\tau_w(x)}{K^2 x^{-\frac{1}{2}}} = 1 \qquad ; \; x < x_1$$

$$= 1 - \tfrac{3}{4}(Rx^{\frac{1}{2}}) \qquad ; \; x > x_1$$

This type of solution may be sufficiently accurate for engineering purposes but a much more difficult analysis could probably include terms of higher order.

The frictional heating term was neglected in the solution of the energy equation but the method of Bernard Le Fur (Ref. 19) in which the frictional heating term is included, could be used for the case of fluid injection.

If improved velocity and temperature distributions near the wall were used, then the accuracy of the method would be improved. A suitable improved velocity distribution is given by Spalding (Ref. 23).

Solutions to the compressible boundary layer are outlined in Appendix B. The simplified skin friction equation for a uniform free stream compares reasonably well with those of Low (Ref. 12) and Lew and Fanucci (Ref. 16).

The calculations are for a Prandtl number of 0.7.

10. Conclusions

The compressible steady laminar boundary layer equations have been solved approximately for arbitrary, pressure gradient, wall temperature distribution and normal velocity through the permeable wall, v_w.

The method is probably sufficiently accurate for practical purposes for negative pressure gradients providing the heat transfer rate and the skin friction are known when the velocity, v_w is zero.

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APPENDIX A

A Partial Transformation of the Momentum Equation

Equation 3.13 is

$$\bar{F}_1 = -(\tfrac{2}{3})^{-\frac{1}{3}} p^{-\frac{2}{3}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \bar{F}_2 \tag{A.1}$$

Rearranging:

$$\bar{F}_2 = -(\tfrac{2}{3})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \cdot p^{\frac{2}{3}} \bar{F}_1 \tag{A.2}$$

The inverse transform of this equation, using the Convolution Theorem, is therefore

$$-\frac{2\tau_w(t)}{\mu\rho} = -(\tfrac{2}{3})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_0^t \frac{(t-t_1)^{-\frac{2}{3}}}{\Gamma(\frac{1}{3})} \left\{ d(u_1^2(t_1)) - \frac{2^{\frac{1}{2}}}{\mu^{3/2}\rho^{\frac{1}{2}}} \tau_w^{\frac{1}{2}}(t_1) v_w(t_1)\, dt_1 \right\}.$$

Hence

$$\frac{2\tau_w(x)}{\mu\rho} = (\tfrac{2}{3})^{\frac{1}{3}} \frac{1}{\Gamma(\frac{2}{3})} \int_0^x \left(\int_{x_1}^x (2\mu\rho\,\tau_w(z))^{\frac{1}{2}} dz \right)^{-\frac{2}{3}} \left\{ d(u_1^2(x_1)) - \frac{2}{\mu} \tau_w(x_1) v_w(x_1) dx_1 \right\}.$$

On rearranging this equation then

$$\tau_w(x) = \frac{(\rho\mu)^{\frac{2}{3}}}{\Gamma(\frac{2}{3}) 3^{\frac{1}{3}}} \left\{ \frac{1}{2} \int_0^x \left(\int_{x_1}^x \tau_w^{\frac{1}{2}}(z) dz \right)^{-\frac{2}{3}} d(u_1^2(x_1)) - \frac{1}{\mu} \int_0^x \left(\int_{x_1}^x \tau_w^{\frac{1}{2}}(z) dz \right)^{-\frac{2}{3}} v_w(x_1)\, \tau_w(x_1) dx_1 \right\}. \quad \ldots\ldots \tag{A.3}$$

The term involving $u_1^2(x)$ is a 'Stieltjès integral' and it has a value when the free stream velocity, u_1, is a constant. The equation is first used in Section 7.2 as equation 7.13.

APPENDIX B

The Compressible Laminar Boundary Layer with Suction or Injection

The method used in this report has now been extended to the compressible boundary layer by using the paper by Lilley (Ref. 18). The integral equations for the skin friction and the heat transfer, which are equivalent to equations 3.14 and 4.29, are:

$$a_o^2\left[M_1^2(o)+\int_o^x \frac{h_w(z)}{h_1}\,d\,M_1^2(z)\right] = \frac{2}{\mu_o\,\rho_o}\int_o^x \frac{\tau_w(x_1)}{c_w(x_1)}\,m(x_1)^{\frac{2(2\gamma-1)}{\gamma}}\,\rho_w(x_1)v_w(x_1)dx_1 +$$

$$\frac{3^{\frac{1}{3}}.\,2}{\Gamma(\frac{1}{3})(\mu_o\,\rho_o)^{2/3}}\int_o^x \frac{\tau_w(x_1)^{3/2}}{c_w(x_1)^{\frac{1}{2}}}\,m(x_1)^{\frac{3\gamma-2}{\gamma}}\left(\int_{x_1}^x \frac{c_w(z)^{\frac{1}{2}}\,\tau_w(z)^{\frac{1}{2}}\,dz}{m(z)}\right)^{-\frac{1}{3}}dx_1 \quad \ldots\ldots \quad (B.1)$$

and

$$Q_w(x) = -\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})3^{\frac{1}{3}}}\cdot\frac{\sigma^{\frac{2}{3}}\,\rho_w(x).\,\tau_w(x)^{\frac{1}{2}}\,\mu_w}{(\rho_o\,\mu_o)^{\frac{1}{3}}\,c_w(x)^{\frac{1}{2}}}\,m(x)\int_o^x\left(\int_{x_1}^x \frac{c_w(z)^{\frac{1}{2}}\tau_w(z)^{\frac{1}{2}}\,dz}{m(z)}\right)^{-\frac{2}{3}}$$

$$\frac{v_w(x_1)\,Q_w(x_1)\,c_w(x_1)^{\frac{1}{2}}\,dx_1}{\mu_w\;\tau_w(x_1)^{\frac{1}{2}}\,m(x_1)} - \frac{3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}\,(\rho_o\,\mu_o)^{\frac{1}{3}}\,\frac{\tau_w(x)^{\frac{1}{2}}\,\rho_w(x)\,m(x)\,\mu_w}{\sigma^{\frac{2}{3}}\,c_w(x)^{\frac{1}{2}}\rho_o\qquad\mu_o}$$

$$\int_o^x\left(\int_{x_1}^x \frac{c_w(z)^{\frac{1}{2}}\,\tau_w(z)^{\frac{1}{2}}\,dz}{m(z)}\right)^{-\frac{1}{3}} d\left(h_{w_o}(x_1)-h_w(x_1)\right) \quad \ldots\ldots \quad (B.3)$$

$$m(x) = \left(1+\frac{\gamma-1}{2}.\,M_1(x)^2\right)^{\frac{\gamma}{2(\gamma-1)}} \quad (B.4)$$

M is the Mach number, $c_w(x)$ is $\left(\frac{\mu_w\,T_o}{\mu_o\,T_w}\right)$ and h is the stagnation enthalpy.

h_{w_o} is the wall enthalpy at zero heat transfer.

Convenient dimensionless forms of these equations are obtained by introducing f_w and f''_w which are defined:

$$f_w = -2\rho_w v_w \left(\frac{x}{u_a \rho_a \mu_a}\right)^{\frac{1}{2}} \quad \text{and } f''_w = \tau_w \left(\frac{x}{\rho_a u_a^3 \mu_a}\right)^{\frac{1}{2}} \qquad \ldots\ldots \quad \text{(B.5)}$$

The suffix 'a' denotes an arbitrary constant reference condition.

Equations B.1 and B.2 become, (when $\gamma = 1.4$)

$$\frac{M_1^2(o)}{M_a^2} + \int_o^x \frac{i_w(z)}{h_1} \cdot \frac{dM_1^2(z)}{M_a^2} = -\int_o^x \frac{f''_w(x_1)\, f_w(x_1)}{x_1} \left(\frac{i_a}{i_w(x_1)}\right)^{\omega-1} \left(\frac{5+M_1^2(x_1)}{5+M_a^2}\right)^{9/2} dx_1 + \frac{3^{\frac{1}{3}}.\,2}{\Gamma(\frac{1}{3})} \int_o^x \frac{f''_w(x_1)}{x_1^{\frac{3}{4}}} \left(\frac{i_a}{i_w(x_1)}\right)^{\frac{\omega-1}{2}} \left(\frac{5+M_1^2(x_1)}{5+M_a^2}\right)^{11/4}$$

$$\left(\int_{x_1}^x \frac{f''_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \left(\frac{i_a}{i_w(z)}\right)^{\frac{1-\omega}{2}} \left(\frac{5+M_a^2}{5+M_1^2(z)}\right)^{7/4} dz\right)^{-\frac{1}{3}} dx_1 \qquad \ldots\ldots \quad \text{(B.6)}$$

$$\frac{Q_w(x)}{(h_w(+o) - h_{w_o}(+o))} \sqrt{\frac{x}{\rho_a \mu_a u_a}} = -\frac{3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})\,\sigma^{\frac{2}{3}}} \left(\frac{i_a}{i_w(x)}\right)^{\frac{1-\omega}{2}} f''_w(x)^{\frac{1}{2}}\ m_a(x)$$

$$\int_o^x \left(\frac{1}{x^{\frac{3}{4}}} \int_{x_1}^x \frac{f''_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} \left(\frac{i_a}{i_w(z)}\right)^{\frac{1-\omega}{2}} m_a(z)\, dz\right)^{-\frac{1}{3}} \frac{d(h_w(x_1) - h_{w_o}(x_1))}{(h_w(+o) - h_{w_o}(+o))}$$

$$+ \frac{\Gamma(\frac{2}{3})}{2\,\Gamma(\frac{1}{3})} \cdot 3^{\frac{1}{3}} \left(\frac{i_w(x)}{i_a}\right)^{\frac{\omega-1}{2}} x^{\frac{1}{4}}\, \sigma^{\frac{2}{3}}\, f''_w(x)^{\frac{1}{2}}\, m_a(x). \int_o^x \left(\int_{x_1}^x \left(\frac{i_w(z)}{i_a}\right)^{\frac{\omega-1}{2}} \frac{f''_w(z)^{\frac{1}{2}}}{z^{\frac{1}{4}}} m_a(z)\, dz\right)^{-\frac{2}{3}} \left\{\sqrt{\frac{x_1}{\rho_a u_a \mu_a}} \cdot \frac{Q_w(x_1)}{(h_w(+o) - h_{w_o}(+o))}\right\} \frac{f_w(x_1)}{f''_w(x_1)^{\frac{1}{2}} x_1^{\frac{3}{4}}}$$

$$m_a(x_1)^{-1} \left(\frac{i_w(x_1)}{i_a}\right)^{\frac{1-\omega}{2}} dx_1 \qquad \text{(B.7)}$$

where $m_a(x) = \left(\dfrac{5 + M_a^2}{5 + M_1(x)^2}\right)^{7/4}$. i is the enthalpy $(h = i + \frac{u^2}{2})$

It is assumed that $\mu \sim T^{\omega}$ where ω is a constant.

Equations B.6 and B.7 may be solved by methods similar to those used in this report for the incompressible flow together with the modifications used in Lilley's paper.

Flat plate with zero pressure gradient and constant wall enthalpy

Considering the velocity $v_w(x)$ to vary as $\frac{1}{\sqrt{x}}$ and using the approximate method of Section 7.2 then the equation B.6 reduces to

$$0.211 + 2.15\, f_w\, f_w'' \left(\frac{i_1}{i_w}\right)^{\omega-1} = f_w''^{\,4/3} \left(\frac{i_1}{i_w}\right)^{\frac{2}{3}(\omega-1)} \qquad \text{(B.8)}$$

The energy equation reduces to

$$k_h\, Re^{\frac{1}{2}} = (k_h\, Re^{\frac{1}{2}})_{v_w(x)=0} \left(\frac{1}{1 - \dfrac{E\, f_w}{f_w''^{\frac{1}{3}}} \left(\dfrac{i_w}{i_1}\right)^{\frac{1-\omega}{3}}} \right) \qquad \text{(B.9)}$$

where $E = \dfrac{\Gamma(\frac{1}{3})\ \sigma^{\frac{2}{3}}}{2^{\frac{1}{3}}\ 3^{\frac{2}{3}}}$ and

k_h is the Stanton heat transfer coefficient given by $k_h = \dfrac{-Q_w(x)}{\rho_a\, u_a (h_w - h_{w_o})}$.

The compressible equations reduced to the incompressible form when the wall enthalpy is constant and the free stream is uniform

$(Q_w)_c$, $(f_w)_c$ and $(f_w'')_c$ will be defined by the following equations:-

$$Q_w = (Q_w)_c \left(\frac{i_w}{i_1}\right)^{\frac{\omega-1}{2}}, \quad f_w = (f_w)_c \left(\frac{i_w}{i_1}\right)^{\frac{\omega-1}{2}}, \quad f_w'' = (f_w'')_c \left(\frac{i_w}{i_1}\right)^{\frac{\omega-1}{2}} \qquad \ldots\ldots \text{(B.10)}$$

When $\left(\frac{i_w}{i_1}\right)$ and M_1 are constant then equations B.6 and B.7 reduce to:

$$1 = -\int_0^x \frac{(f''_w(x_1))_c\,(f_w(x_1))_c\,dx_1}{x_1} + \frac{3^{\frac{1}{3}}.\,2}{\Gamma(\frac{1}{3})}\int_0^x \frac{(f''_w(x_1))_c^{3/2}}{x_1^{\frac{3}{4}}}$$

$$\left(\int_{x_1}^x \frac{(f''_w(z))_c^{\frac{1}{2}}}{z^{\frac{1}{4}}}\,dz\right)^{-\frac{1}{3}} dx_1 \qquad \text{(B.11)}$$

and

$$\frac{(Q_w)_c}{(h_w(+o) - h_{w_o}(+o))}\sqrt{\frac{x}{\rho_1\,\mu_1\,u_1}} = -\frac{3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})\,\sigma^{\frac{2}{3}}}\,(f''_w)_c^{\frac{1}{2}}\;x^{\frac{1}{4}}\left(\int_0^x \frac{(f''_w)_c^{\frac{1}{2}}}{z^{\frac{1}{4}}}\,dz\right)^{-\frac{1}{3}}$$

$$+ \frac{\Gamma(\frac{2}{3})}{2\Gamma(\frac{1}{3})3^{\frac{1}{3}}}\;x^{\frac{1}{4}}\,\sigma^{\frac{2}{3}}\,(f''_w)_c^{\frac{1}{2}}\int_0^x\left(\int_{x_1}^x \frac{(f''_w)_c^{\frac{1}{2}}}{z^{\frac{1}{4}}}\,dz\right)^{-\frac{2}{3}}\left\{\frac{(Q_w)_c}{(h_w(+o)-h_{w_o}(+o))}\right.$$

$$\left.\sqrt{\frac{x}{\rho_1\,\mu_1\,u_1}}\right\}\;\frac{(f_w)_c}{(f''_w)_c^{\frac{1}{2}}\,x_1^{\frac{3}{4}}}\;dx_1 \qquad \text{(B.12)}$$

These are equivalent to the incompressible equations when m = 0. Therefore the solutions may be written directly from the main text of this report.

(a) When the velocity $v_w(x)$ is proportional to $\frac{1}{\sqrt{x}}$ then

$$(f''_w)_c^{4/3} = 0.211 + 2.15\,(f_w)_c\,(f''_w)_c \qquad \text{(B.13)}$$

and

$$k_h\,Re^{\frac{1}{2}} = (k_h\,Re^{\frac{1}{2}})_{v_w(x)=0}\left(\frac{1}{1 - \dfrac{E(f_w)_c}{(f''_w)_c^{\frac{1}{3}}}}\right) \qquad \text{(B.14)}$$

(These are the same as equations B.8 and B.9).

Equation B.13 compares favourably with the solution of the compressible boundary layer by Low (Ref. 12). In Fig. 3 the curve of Donoughe and Livingood for m = 0 is identical with Low's curve and the other curve for m = 0 is identical with that of equation B.13, except that f_w and f''_w are now replaced by $(f_w)_c$ and $(f''_w)_c$.

(b) When the velocity v_w is constant then, from Section 8, equation 8.24

$$\frac{\tau_w(x)}{(\tau_w(x))_{v_w(x)=0}} = 1 + 0.982\,(f_w)_c = 1 + 0.982\, f_w \cdot \left(\frac{i_w}{i_1}\right)^{\frac{1-\omega}{2}} \quad \text{(B.15)}$$

or equation 8.25

$$= 1 + 1.25\,(f_w)_c \quad \text{(B.16)}$$

Lew and Fanucci (Ref. 16) present a curve for the skin friction with uniform suction for the compressible boundary layer.

$\left(\frac{\tau_w}{\rho_w v_w u}\right)$ is plotted against $\sqrt{\frac{2\xi}{C}} \cdot \left(\frac{T_1}{T_w}\right)$ where $\xi = \left(-\frac{v_w}{u_1}\right)^2 \left(\frac{u_1 x}{\nu_1}\right)$

and $C = \frac{\mu}{\mu_1} \cdot \frac{T_1}{T}$. These terms may be compared with those in this paper if C is assumed equal to $\left(\frac{T_w}{T_1}\right)^{\omega-1}$.

Then $-\frac{\tau_w}{\rho_w v_w u_1} = \frac{2\,(f''_w)_c}{(f_w)_c}$ and $\sqrt{\frac{2\xi}{C}}\left(\frac{T_1}{T_w}\right) = \frac{\sqrt{2}}{2} \cdot (f_w)_c$.

Equations B.15 and B.16 are compared with Lew and Fanucci's results in Fig. 8.

TABLE 1

m	f_w	f''_w From Equ. 7.8	f''_w From Ref. 6*	% Error
1.0	0	1.136	1.233	8
	-0.5	.784	.969	19
	-1.0	.586	.756	22
0.5	0	.830	.900	8
	-0.5	.554	.697	20
	-1.0	.408	.534	24

TABLE 2

m	f_w	f''_w From Equ. 7.12	f''_w From Ref.6*	% Error
1.0	0	1.233	1.233	-
	-0.5	0.881	.969	9
	-1.0	.683	.757	10
0.5	0	.900	.900	-
	-0.5	.624	.697	10
	-1.0	.478	.534	11

* The Results of Donoughe and Livingood

TABLE 3

f_w	m	ϵ	nu $Re^{-\frac{1}{2}}$ From Equ. 7.42	nu $Re^{-\frac{1}{2}}$ From Ref. 6	% Error
-0.5	0	0	.133	.166	-20
		0.5	.226	.261	-13
		1.0	.284	.321	-12
	0.5	0	.240	.259	-7
		0.5	.361	.383	-6
		1.0	.445	.471	-6
	1.0	0	.279	.293	-5
		0.5	.393	.413	-5
		1.0	.478	.503	-5
-1.0	0	0	[.040	.052	-23
		0.5	.081	.105	-23
		1.0	.108]	.138	-22
	0.5	0	[.152]	.139	10
		0.5	.251	.253	-1
		1.0	.318	.331	-4
	1.0	0	[.177]	.146	21
		0.5	.267	.255	5
		1.0	.336	.336	0

The Brackets, [ ], denote that $\left| \dfrac{D\, f_w}{f_w''^{\frac{1}{3}}} \right| > 1$.

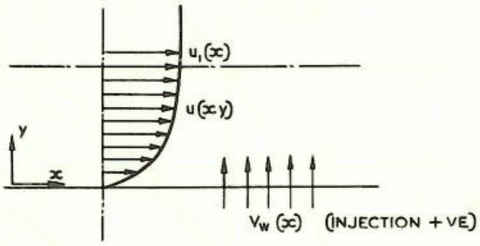

FIG. 1.

ε IS WALL TEMPERATURE-GRADIENT PARAMETER
nu IS LOCAL NUSSELT NUMBER
f_w IS DIMENSIONLESS FORM OF FLOW THROUGH POROUS WALL

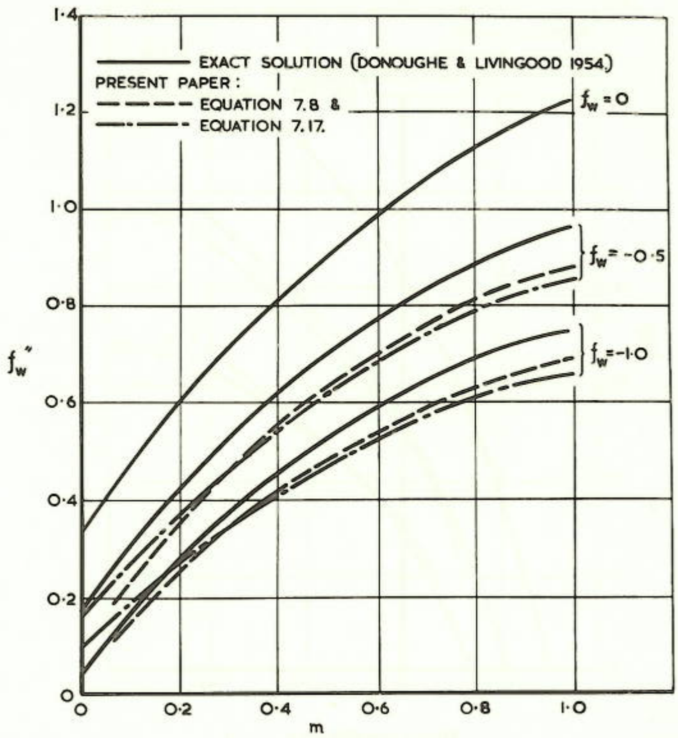

FIG.2. THE VARIATION OF THE SKIN FRICTION PARAMETER, f_w'' WITH THE EULER NUMBER, m

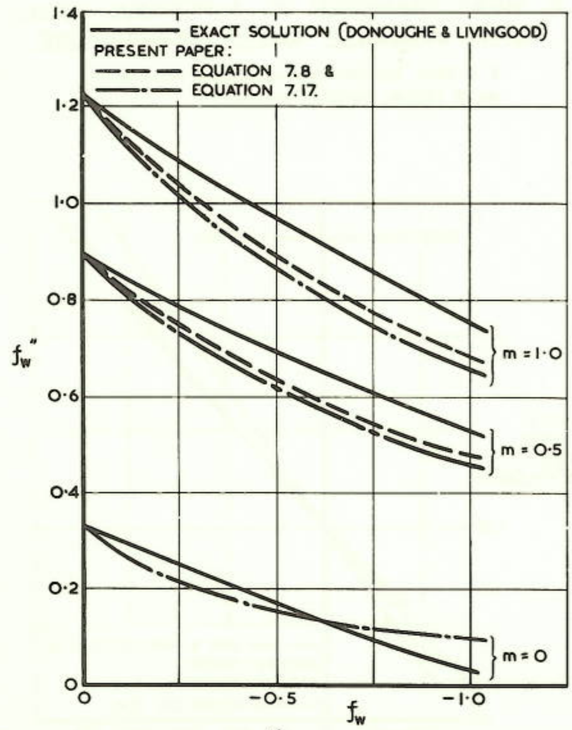

FIG. 3. THE EFFECT OF f_w AND THE EULER NUMBER, m, ON THE SKIN FRICTION PARAMETER f_w''

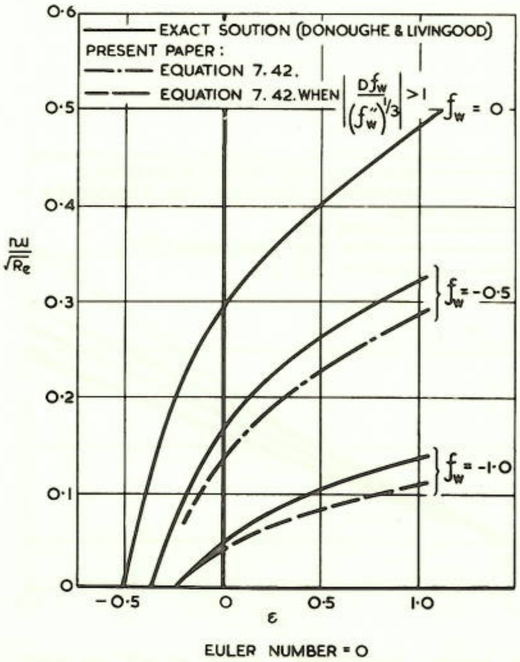

FIG. 4. HEAT TRANSFER TO A POROUS WALL WITH VARIABLE WALL TEMPERATURE

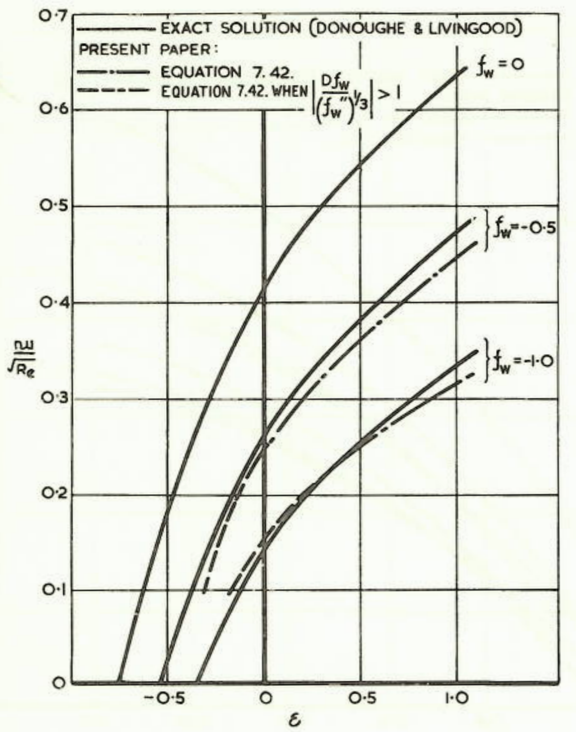

FIG. 5. HEAT TRANSFER TO A POROUS WALL WITH VARIABLE WALL TEMPERATURE

ε IS WALL TEMPERATURE-GRADIENT PARAMETER
nu IS LOCAL NUSSELT NUMBER

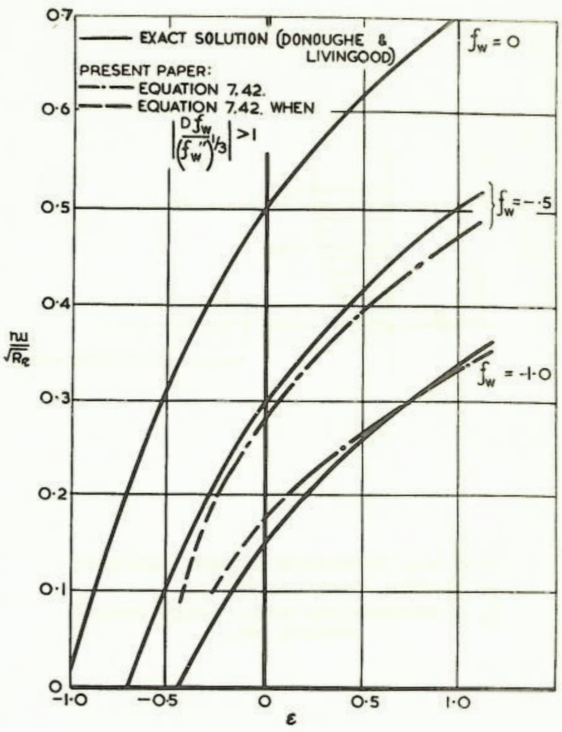

FIG. 6. HEAT TRANSFER TO A POROUS WALL WITH VARIABLE WALL TEMPERATURE

ε IS WALL TEMPERATURE-GRADIENT PARAMETER
nu IS LOCAL NUSSELT NUMBER

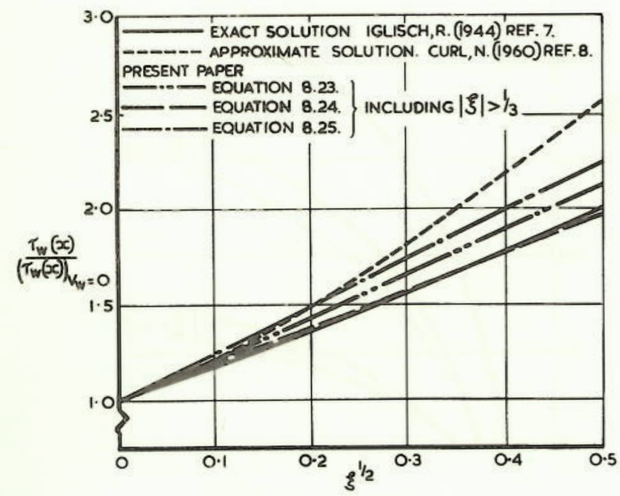

FIG. 7. SKIN FRICTION FOR FLAT PLATE WITH UNIFORM SUCTION

$$\xi = \frac{V_w^2 x}{U_1 \nu}$$

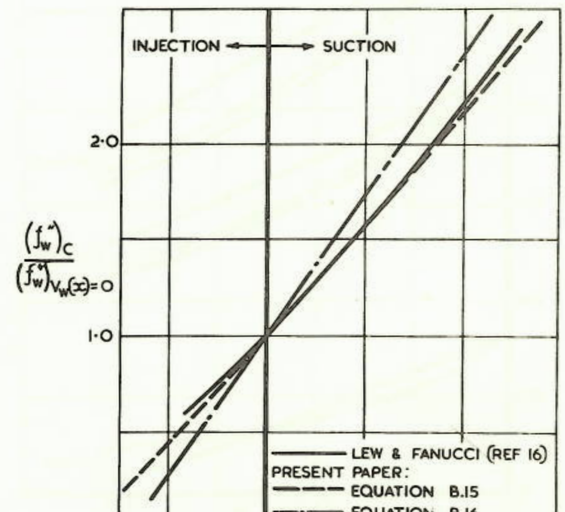

FIG. 8. SKIN FRICTION WITH UNIFORM SUCTION AND INJECTION (COMPRESIBLE SOLUTION)

$$\xi = \frac{V_w^2 x}{u_1 \nu_1}$$